

Evolution of the Quantum Friedmann Universe Featuring Radiation ^a

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Abstract: The classical and quantum models of the Friedmann universe originally filled with a scalar field and radiation have been studied. The radiation has been used to specify a reference frame that makes it possible to remove ambiguities in choosing the time coordinate. Solutions to the Einstein and Schrödinger equations have been studied under the assumption that the rate of scalar-field variation is much less than the rate of universe expansion (contraction). It has been shown that, under certain conditions, the quantum universe can be in quasistationary states. The probability that the universe goes over to states with large quantum numbers owing to the interaction of the scalar and gravitational fields is nonzero. It has been shown that, in the lowest state, the scale factor is on order of the Planck length. The matter- and radiation-energy densities in the Planck era have been computed. The possible scenarios of Universe evolution are discussed.

1. INTRODUCTION

That quantum gravity theory cannot rely on experimental data [1] adds importance to exactly soluble cosmological models. However, the application of basic ideas underlying quantum theory to a system of gravitational and matter fields runs into difficulties of a fundamental character, which do not depend on the choice of a specific model. By way of example, we will consider a homogeneous, isotropic, and closed universe characterized by the Friedmann-Robertson-Walker metric; that is,

$$ds^2 = a^2(\eta) [N^2(\eta) d\eta^2 - d\Omega^2], \quad (1)$$

Here, $N(\eta)$ is a function that specifies the time-reference scale; $a(\eta)$ is a scale factor; $d\Omega^2$ is an interval element on a unit 3-sphere; and η is the parameter that is related to the synchronous proper time t by the differential equation $dt = N a d\eta$. Considering that scalar fields play a fundamental role both in quantum field theory and in the cosmology of the early Universe [2, 3], we assume that, originally, the Universe was filled with matter in the form of a uniform scalar field ϕ . If the field ϕ varies slowly in the early Universe, its potential $V(\phi)$ specifies the vacuum-energy density (cosmological term) and ensures Hubble expansion. Restricting our analysis to the case of minimal coupling between geometry and the scalar field, we represent the action functional in the conventional form

$$S = \int d\eta [\pi_a a' + \pi_\phi \phi' - H], \quad (2)$$

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where a prime denotes differentiation with respect to $d/d\eta$; π_a and π_ϕ are the momenta canonically conjugate with the variables a and ϕ , respectively; and H is the Hamiltonian given by

$$H = \frac{1}{2} N \left[-\pi_a^2 + \frac{2}{a^2} \pi_\phi^2 - a^2 + a^4 V(\phi) \right] \equiv N \mathcal{R}. \quad (3)$$

Here, the variables a and ϕ are taken, respectively, in units of the length $l = \sqrt{2G/3\pi}$ and in units of $\tilde{\phi} = \sqrt{3/8\pi G}$. The function N plays the role of a Lagrange multiplier, and the variation $\delta S/\delta N$ leads to the constraint equation $\mathcal{R} = 0$. The structure of the constraint is such that true dynamical degrees of freedom cannot be singled out explicitly. This creates problems in the interpretation of quantum geometrodynamics [46]. It is commonly thought that the main reason behind such difficulties is that there is no natural way to define a spacetime event in general covariant theories [7]. In the model being considered, the above difficulties are reflected in that the choice of the time variable is ambiguous.

For the choice of the time coordinate to be unambiguous, the model must be supplemented with a coordinate condition. When the coordinate condition is added to the field equations, their solution can be found for a fixed time variable. However, this method of removing ambiguities in specifying the time variable does not solve the problem of a quantum description, because undesirable consequences of this ambiguity in eventual equations cannot be avoided in this way.

In this study, we propose specifying a reference frame with the aid of an additional matter source. This method does not come into conflict with the adopted ideas of the early Universe [2, 3]. At the same time, it enables us to study the evolution of the Universe not only in the semiclassical approximation but also at a purely quantum level.

2. CLASSICAL DESCRIPTION

2.1. Fundamentals of the Model

The ambiguity associated with choosing the time coordinate in (1) will be removed with the aid of a coordinate condition imposed prior to varying the action functional, but its coordinate invariance will be restored [7, 8]. We will choose the coordinate condition in the form $T' = N$, where T is the privileged time coordinate, and include it in the action functional with the aid of a Lagrange multiplier P ; that is,

$$S = \int d\eta \left[\pi_a a' + \pi_\phi \phi' + P T' - \mathcal{H} \right], \quad (4)$$

where

$$\mathcal{H} = N [P + \mathcal{R}] \quad (5)$$

is the new Hamiltonian. The constraint equation reduces to the form

$$P + \mathcal{R} = 0. \quad (6)$$

Integrating the canonical equation $P' = [P, \mathcal{H}] = 0$, we immediately obtain $P = E$, where E is a constant. The full set of equations for the model in question becomes

$$\dot{a}^2 - \frac{a^2}{2} \dot{\phi}^2 + U = E, \quad (7)$$

$$\ddot{\phi} + 2 \frac{\dot{a}}{a} \dot{\phi} + a^2 \frac{dV}{d\phi} = 0, \quad (8)$$

where overdots denote differentiation with respect to T and $U \equiv a^2 -^4 V(\phi)$. Equation (7) represents the Einstein equation for the $(^0_0)$ component, while equation (8) is the equation of motion for the field ϕ . A modification to the Einstein equations that is associated with including the coordinate condition in the action functional is that, on the right-hand side, there additionally arises an energy-momentum tensor $\tilde{T}_0^0 = E/a^4$, $\tilde{T}_1^1 = \tilde{T}_2^2 = \tilde{T}_3^3 = -E/3a^4$, $\tilde{T}_\beta^\alpha = 0$ for $\alpha \neq \beta$ that can be interpreted as the energy-momentum tensor of radiation [9]. The choice of radiation as the matter reference frame is natural for the case in which relativistic matter (electromagnetic radiation, neutrino radiation, etc.) is dominant at the early stage of Universe evolution. If our Universe were described by the model specified by equation (4), it would be possible to relate the above radiation at the present era to cosmic microwave background radiation.

2.2. Solving the Einstein Equations

A feature peculiar to the model in question is that it involves a barrier in the variable a . This barrier, described by the function U , is formed by the interaction of the scalar and gravitational fields. It exists for any form of the scalar-field potential $V(\phi)$ and becomes impenetrable on the side of small a in the limit $V \rightarrow 0$. In just the same way as in inflation models (see [2, 3]), we assume that the rate at which the scalar field ϕ changes is much smaller than the rate of universe evolution, $|\dot{a}/a| \gg |\dot{\phi}|$. In this case, equation (7) can be integrated in a general form. Presented immediately below are explicit solutions in the regions $a \leq a_1$ and $a \geq a_2$, where they are assumed to satisfy the boundary conditions $a(0) = 0$ and $a(t_{in}) = a_2$, respectively; here, a_1 and a_2 are the turning points ($a_1 < a_2$) specified by the condition $U = \epsilon$, and t_{in} is the initial instant of time in the second region. We have

$$a(t) = \left[\frac{1}{2V} \left(1 - \cosh 2\sqrt{V}t \right) + \sqrt{\frac{\epsilon}{V}} \sinh 2\sqrt{V}t \right]^{1/2} \quad (9)$$

for $a \leq a_1$ and

$$a(t) = \left\{ \frac{1}{2V} \left[1 + \sqrt{1 - 4V\epsilon} \cosh 2\sqrt{V}(t - t_{in}) \right] \right\}^{1/2} \quad (10)$$

for $a \geq a_2$. The quantities ϵ and U depend parametrically on ϕ . In the zero-order approximation, the former is given by $\epsilon = E$. The above solution to equation (7) can be refined by taking into account a slow variation of the field ϕ with the aid of the equation

$$-\frac{a^2}{2} \dot{\phi}^2 + \epsilon = E, \quad (11)$$

where $\epsilon(\phi)$ stands for a potential term, which is bounded by the inequality $E \leq \epsilon \leq 1/4V$. The case of $\epsilon > 1/4V$, which corresponds to an infinite motion, will not be considered in this study. From equations (8) and (11), it follows that, in general, a change in the potential $V(\phi)$ entails a change in the quantity $\epsilon(\phi)$.

The solution in (9) describes the universe expanding from the point of the initial cosmological singularity to the maximum possible value of a_1 achieved at the instant

$t_m = \frac{1}{4\sqrt{V}} \ln \left(\frac{1+2\sqrt{V\epsilon}}{1-2\sqrt{V\epsilon}} \right)$; after that, the expansion gives way to contraction, and the universe collapses by the instant $t_c = 2 t_m$. For $2\sqrt{V}t \ll 1$, the solution in (9) takes the form

$$a(t) \simeq [2\sqrt{\epsilon}t]^{1/2}. \quad (12)$$

It is independent of V and describes the evolution of the universe that is dominated by radiation [9] and which expands in the de Sitter mode from the point $a = a_2$. In the extreme case of $\epsilon = 0$, where there is no radiation, the region $a \leq a_1$ contracts to the point $a = 0$, and the expansion can proceed only from the point $a = a_2$. Since the region $a < a_2$ cannot be treated in terms of classical theory, it is assumed that the classical spacetime with $a = a_2$ is formed as the result of a tunnel transition from "nothing" taken to mean some quantum state of the protouniverse (see, for example, [2, 3, 10, 11]). If, originally, the universe was filled not only with matter but also with radiation, it can undergo evolution in the region $a \leq a_1$ as well. In the general theory of relativity, the solutions in (9) and (10) describe two independent scenarios of the evolution. The inclusion of the mechanism of quantum tunneling through the barrier U requires a joint analysis of these scenarios. It is then legitimate to consider the probabilities of finding the universe in each of the classically accessible regions.

The evolution of the universe depends on the initial distribution of the classical field ϕ and its subsequent behavior as a function of time. The chaotic-inflation scenario [3], which is realized in the region $a > a_2$, is described by equations (7) and (8) as applied to the case specified by the inequalities $(\frac{d\ln V}{d\phi})^2 \ll 1$, $V \gg |\frac{1}{a^2} - \frac{\epsilon}{a^4}|$, and $\frac{1}{a^2}|\ddot{\phi}| \ll |\frac{dV}{d\phi}|$. In the model where the scalar-field potential V is taken to be proportional to ϕ^n , the chaotic-inflation process proceeds between scalar-field values greatly exceeding a level of $\frac{n}{3\sqrt{2}}$ (initial stage) and those achieving this level (final stage). In this approach, radiation has virtually no effect on the degree of inflation, and the scalar field represents the field of an inflaton. The de Sitter regime of inflation persists as long as the potential $V(\phi(t))$ varies rather slowly with time. From equations (8) and (11), it follows that the inequality $\dot{V} + \dot{\epsilon}/a^4 < 0$ holds in the expanding universe ($\dot{a} > 0$). If the potential V increases with time, the quantity ϵ is bound to decrease. But if V decreases, ϵ can increase, and the rate of this increase is higher for greater a . We will now estimate ϵ by using the relation $\epsilon \simeq \tilde{T}_0^0 a^4$. In our Universe, with $a \sim 10^{28}$ cm, the main contribution to the radiation-energy density comes from cosmic microwave background radiation with energy density $\rho_\gamma^0 \sim 10^{-10}$ GeV/cm³. Setting $\tilde{T}_0^0 = \rho_\gamma^0$, we find that, at the present era, the result is $\epsilon = \epsilon_\gamma \sim 10^{117}$. In the early Universe, the scale parameter is $a \sim 10^{-33}$ cm, while the energy density \tilde{T}_0^0 is on the order of the Planck value. On this basis, it can be found that $\epsilon \sim 1$ corresponds to that era. It follows that ϵ increased in the evolution process. This increase can be explained by a considerable redistribution of energy between the scalar field and radiation at the initial stage of Universe existence. Quantum theory is able to account for this phenomenon in a natural way (see below). In the region $a > a_2$, the possible variation of ϵ with time does not affect the quasiexponential expansion of the universe, because the inflation stage terminates in a rather short time interval of $t \sim 10^{-37}$ s [3], and the evolution process is then determined by other factors (particle production, heating, etc.). In the region $a < a_1$, the role of the increase in ϵ with decreasing V may prove substantial. In principle, the dependence of ϵ on $\phi(t)$ makes it possible to provide the missing power in the t dependence of a and to solve the problem of the Universe size. In order to demonstrate this explicitly, we assume that, up to the present time

t_0 , the Universe has expanded according to the law specified by (12) [2]. We then have $a(t_0)/a(t_p) = \left(\sqrt{\epsilon_0/\epsilon_p} t_0/t_p\right)^{1/2}$, where $\epsilon_0 = \epsilon(\phi(t_0))$, $\epsilon_p = \epsilon(\phi(t_p))$, and t_p is the Planck time. The ratio of ϵ_0 and ϵ_p can be estimated as $\epsilon_0/\epsilon_p \sim V_p/V_0$, where $V_p = V(\phi(t_p))$ and $V_0 = V(\phi(t_0))$. Assuming that, in the Planck era, V_p is on the order of the Planck energy density and that V_0 is on the order of the mean matter-energy density at the present era, $\rho_0 = 10^{-5}\text{GeV}/\text{cm}^3$, we find that the value of $a(t_0) \sim 10^{28}$ cm corresponds to $a(t_p) \sim 10^{-33}$ cm.

3. QUANTIZATION

3.1. Schrödinger Equation

In quantum theory, the constraint equation (6) comes to be a constraint on the wave function that describes the universe filled with a scalar field and radiation. Replacing the canonically conjugate variables involved in equation (7) by the operators $\hat{a} = a \times$, $\hat{\pi}_a = -i \partial_a$, $\hat{\phi} = \phi \times$, $\hat{\pi}_\phi = -i \partial_\phi$, and $\hat{P} = -i \partial_T$, we find that the state vector $\langle a, \phi | \Psi(T) \rangle$ satisfies the equation $\langle a, \phi | \Psi(T) \rangle$

$$2i \partial_T |\Psi(T)\rangle = \left[\partial_a^2 - \frac{2}{a^2} \partial_\phi^2 - U \right] |\Psi(T)\rangle. \quad (13)$$

where the order parameter is assumed to be zero [5, 10-13]. Equation (13) represents an analog of the Schrödinger equation with a Hamiltonian independent of the time variable T . The momentum \hat{P} associated with radiation appears linearly in equation (13). We can introduce a positive definite scalar product $\langle \Psi | \Psi \rangle < \infty$ and specify the norm of a state [8, 11]. This makes it possible to define a Hilbert space of physical states and to construct quantum mechanics for the universe model being considered.

A partial solution to equation (13) has the form

$$|\Psi(T)\rangle = |\psi\rangle \exp \left\{ \frac{i}{2} E (T - T_0) \right\}, \quad (14)$$

where the state ψ satisfies the time-independent equation

$$\left(-\partial_a^2 + \frac{2}{a^2} \partial_\phi^2 + U - E \right) |\psi\rangle = 0. \quad (15)$$

The quantity E is arbitrary in the general theory of relativity, but, in quantum theory, it is quantized in accordance with solutions to equation (15).

3.2. Quasistationary States

In considering the quantum case, we assume that, at the initial stage, the motions occurring in the system under study can be separated into two types: a slow variation of the scalar field, in which case the operator $(2/a^2)\partial_\phi^2$ can be treated as a perturbation, and fast changes in geometry. This assumption is a quantum analog of the adiabaticity hypothesis, which leads, in the zero-order approximation, to the solutions given by (9) and (10). In quantum theory, the problem being considered reduces to solving the equation

$$[\partial_a^2 - U + \epsilon_n(\phi)] |\varphi_n\rangle = 0. \quad (16)$$

The wave functions φ_n and the eigenvalues ϵ_n , which depend on ϕ parametrically, describe the evolution of the universe for very slow variations of the potential V associated with the field ϕ (more specifically, under the condition $|d \ln V / d\phi| \ll 1$). In order to take into account the variations of the field ϕ , we can represent ψ as an expansion in terms of the states φ_n and integrate then equation (15). The quantum number n of the system unperturbed by the operator $(2/a^2)\partial_\phi^2$ will be a good quantum number for the universe in the perturbed state ψ as well.

We will further consider solutions to equation (16), allowing for the possible boundary conditions. For $\epsilon_n \leq 1/4V$ the classically accessible regions $a \leq a_1$ and $a \geq a_2$ are bounded by the turning points a_1 and a_2 , which are now dependent on the state of the quantum system. In the region $a > a_2$, a general solution has the form of a superposition of converging and diverging waves. In the Wentzel-Kramers-Brillouin (WKB) approximation, we can write

$$\varphi_n = \frac{1}{(\epsilon_n - U)^{1/4}} \left\{ C_1 e^{i \int_{a_2}^a \sqrt{\epsilon_n - U} da - \frac{i\pi}{4}} + C_2 e^{-i \int_{a_2}^a \sqrt{\epsilon_n - U} da + \frac{i\pi}{4}} \right\}, \quad (17)$$

where C_1 is the amplitude of an "incident" wave describing the universe whose scale factor decreases, while C_2 is the amplitude of the wave "traveling" toward greater values of a and describing the expanding universe. For the extreme case of $\epsilon_n = 0$, which corresponds to the radiation-free universe with an undetermined time variable, the wave function in the form (17) was studied by many authors (see, for example, [3, 5, 10-13]). It coincides with the Vilenkin wave function [10] at $C_1 = 0$ and generalizes the Hartle-Hawking wave function [11] to the case of $\epsilon_n \neq 0$.

If we consider a universe formed at an instant separated by a comparatively large time interval $[-\infty < (T - T_0) \leq 0]$ from the commencement of observation, the boundary condition $C_1 = 0$ will imply that the diverging wave corresponding to a quasistationary state [14] is singled out from the superposition in (17). No situation that is physically realizable can exactly correspond to the requirement $C_1 = 0$ for all instants of time because, in that case, the process that leads to the formation of a quasistationary state and which involves converging waves would be eliminated from the analysis. According to the general concepts of quantum theory [14, 15], a quasistationary state can be implemented approximately by requiring that the region where the asymptotic form (17) with $C_1 = 0$ is realized be bounded by the condition $a \leq a_{max}$, in which cases φ_n is set to zero for $a > a_{max}$, $a_{max} \sim \sqrt{\epsilon_n} T$ being some boundary value of the scale factor. That the instants of time that satisfy the conditions $T < T_0$ and $T > a_{max}/\sqrt{\epsilon_n}$ are excluded from the analysis is physically justified because this makes it possible to avoid speculations about the properties of the scalar field and radiation in the Universe at times that are practically inaccessible to observation and for which there are no reliable hints from high-energy physics.

The possibility of quantum tunneling through the region $a_1 \leq a \leq a_2$ of the potential barrier results in that stationary states cannot be realized in the region $a \leq a_1$. If, however, the potential $V(\phi)$ is sufficiently small, quasistationary states whose lifetime is much greater than the Planck time can exist in the region $a \leq a_1$. The probability Γ_n of the decay of the universe occurring in a given quasistationary state φ_n can be found by requiring that the wave function in (17) satisfy the radiation condition $C_1 = 0$, which selects the discrete complex values $\tilde{\epsilon}_n = \epsilon_n + i\Gamma_n$ [14].

Table 1: Values of ϵ_n as computed in the WKB approximation on the basis of (19) and within perturbation theory (PT) for various values of the potential V (here, $U_{max} = 1/4V$ is the barrier height; $\Delta a = a_2 - a_1$ is the barrier width; and Σ is the number of levels at given V), along with the decay-probability values Γ_n as given by (18) under the same conditions

n	V	$\epsilon_n(\text{WKB})$	$\epsilon_n(\text{PT})$	Γ_n	U_{max}	Δa	Σ
0	0.08	2.62	2.63	0.31	3.125	1.03	1
	0.05	2.79	2.79	0.006	5	2.25	1
	0.03	2.89	2.88	2×10^{-6}	8.33	3.70	2
	0.02	2.93	2.92	7×10^{-11}	12.5	5.08	3
	0.01	2.97	2.96	10^{-24}	25	8.05	6
1	0.03	6.34	6.35	0.01	8.33	2.06	2
	0.02	6.59	6.59	10^{-6}	12.5	3.70	3
	0.01	6.80	6.80	10^{-19}	25	6.92	6
2	0.02	9.88	9.94	0.003	12.5	2.34	3
	0.01	10.51	10.51	10^{-15}	25	5.93	6

We further impose the boundary condition $\varphi_n|_{a=0} = 0$ on the wave function φ_n . For the case of $\Gamma_n \ll \epsilon_n$, we then obtain

$$\Gamma_n = 2 \left[\int_0^{a_1} \frac{da}{\sqrt{\epsilon_n - U}} \right]^{-1} \exp \left\{ -2 \int_{a_1}^{a_2} \sqrt{U - \epsilon_n} da \right\}, \quad (18)$$

where ϵ_n is determined from the equation

$$\int_0^{a_1} \sqrt{\epsilon_n - U} da = \pi \left(n + \frac{3}{4} \right). \quad (19)$$

In the extreme case of $V = 0$, equation (19) can easily be integrated, which yields $\epsilon_n|_{V=0} \equiv \epsilon_n^{(0)} = 4n + 3$; that is, we can see that, for all values of n , $\epsilon_n^{(0)}$ coincides with the energy of an isotropic oscillator with zero orbital angular momentum [16, 17]. According to (19), the first level (that at $\epsilon_0 \sim 3$) emerges at $V \sim 0.08$.

At small V , equation (19) leads to ϵ_n values coincident with those that are obtained directly from equation (16) by perturbation theory in $a^4 V(\phi)$. The table displays ϵ_n values calculated by perturbation theory and in the WKB approximation [that is, with the aid of equation (19)]. Also presented in this table are the decay-probability values Γ_n computed for various potentials V . Since $\Gamma_n \ll \text{Re} \tilde{\epsilon}_n$, the decay-probability values Γ_n found on the basis of (18) are expected to be close to true values for small n as well.

We note that the smaller the value of $a^4 V$ at given ϵ_n , the higher and the broader is the potential barrier U and, hence, the smaller is the decay probability Γ_n . If some state φ_n is characterized by a small value of Γ_n , the possibility that this state decays can be disregarded over the decay time $\tau = 1/\Gamma_n$, so that this state can be considered to be stationary in this limit. This corresponds to defining a quasistationary state as that which takes the place of a stationary state when the probability of its decay becomes nonzero [14].

In describing the universe on the basis of equation (15), the process of universe production from "nothing" in the radiation-free model ($E = 0$) [2, 10, 12, 13] is replaced by quantum tunneling from a quasistationary state with a definite (complex) value of E . It can easily be seen that the solution given by (17) describes the de Sitter regime of expansion according to (10). In order to demonstrate this explicitly, we note that, in the WKB approximation, we have $-i\partial_a\varphi_n \approx -\sqrt{\epsilon_n - U}\varphi_n$; that is, the classical momentum is given by $\pi_a = -\dot{a} = -\sqrt{\epsilon_n - U}$, whence we obtain equation (7) in approximation $|\dot{a}/a| \gg |\dot{\phi}|$.

In quantum models not featuring radiation, the DeWitt-Wheeler equation [5, 18] determines the time-independent wave function of the universe. This leads to well-known difficulties in interpreting this function and in comparing results obtained on its basis with the observed evolution of our Universe [3, 6]. If, however, the case of $E = 0$ is considered as the limit to which the model with a privileged reference frame reduces when $E \rightarrow 0$, we can also speak about the time evolution of the Universe free from radiation [19].

3.3. Dynamics in the Prebarrier Region

By studying inflationary scenarios of the evolution of a universe filled with a scalar field, it was revealed that a "realistic" potential V must decrease with time [10]. With decreasing V , the number of quantum states in which the universe can occur increases, while the decay probabilities decrease sharply (see table). The first instants of the existence of the universe are especially favorable for its tunneling through the potential barrier U .

A quasistationary state φ_n takes the place of the stationary state whose wave function in the region $a < a_1$; is close to the wave function $|n\rangle$ of the state unperturbed by the interaction a^4V . In the approximation of a slowly varying field ϕ , transitions in the system being studied can be considered as those that occur between the states $|n\rangle$ and which are induced by the interaction a^4V . Since this interaction modifies the physical properties of the system, a finite number of its levels and their nonzero widths must be taken into account in calculating the probabilities W_{nm} of the $m(T_0) \rightarrow n(T)$ transitions. As a result, we arrive at

$$W_{nm} \approx |\langle n|\mathcal{U}_I(T, T_0)|m\rangle|^2 \exp\{-\Gamma_n\Delta T\}, \quad (20)$$

where $\Delta T = T - T_0$ and \mathcal{U}_I is the evolution operator in the interaction representation [20]. Adiabaticity in the field ϕ enables us to consider specific transitions in the time interval ΔT that correspond to a given value of $V(\phi)$.

The figure displays the total probability of universe decay, $W_{dec} = 1 - (W_{00} + W_{10})$, and the quantity W_{10} as calculated at $V = 0.03$, in which case there are only two levels in the system. It can be seen that, over the time interval $\Delta T \lesssim 50$, the transitions in the system predominate and only for $\Delta T \sim 100$ the probability that the universe tunnels through the barrier becomes commensurate with the probability that it undergoes the $0 \rightarrow 1$ transition in the prebarrier region.

Since the rate at which the level width Γ_n tends to zero is greater than the rate at which the potential decreases, the reduction of V with time results in that transitions become much more probable than tunnel decays, in which case the former fully determine the evolution of the quantum universe in the prebarrier region. If the universe has not tunneled through the barrier before the potential V of the field ϕ decreases to a value less

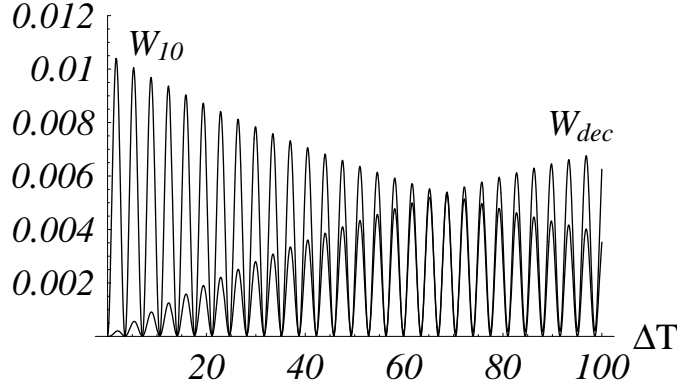


Figure 1: Probabilities W_{10} and W_{dec} versus the time interval $\Delta T = T - T_0$ at the parameter values of $V = 0.03$, $\epsilon_0^{(0)} = 3$, $\epsilon_1^{(0)} = 7$, $\Gamma_0 = 2 \times 10^{-6}$, and $\Gamma_1 = 10^{-2}$.

than 0.01, a sufficiently large number of levels such that the probabilities of decays from them can be neglected are formed in it. Assuming that the amplitudes of transitions over the time interval ΔT are small, $|V \langle n | a^4 | m \rangle| \Delta T \ll 1$, we then find that

$$\frac{W_{n+1,n}}{W_{n-1,n}} > 1, \quad \frac{W_{n+1,n}}{W_{n+2,n}} > 1, \quad \frac{W_{n+2,n}}{W_{n-2,n}} > 1, \quad \frac{W_{n-1,n}}{W_{n+2,n}} > 1, \quad (21)$$

that is, the $n \rightarrow n + 1$ transition is more probable than the $n \rightarrow n - 1$ and $n \rightarrow n + 2$ transitions. This means that the quantum universe can undergo transitions to ever higher levels with a nonzero probability. It is well known that the oscillator amplitude is quantized according to the condition $\bar{a} \sim \sqrt{n}$; therefore, it can be concluded that the characteristic size \bar{a} of the universe that did not undergo a tunnel transition increases as it is excited to higher levels.

3.4. Parameters of the Early Universe

In the adiabatic approximation, the expectation value \bar{a} for the universe occurring in the lowest state φ_0 is given by

$$\bar{a} \approx \langle \varphi_0 | a | \varphi_0 \rangle = \frac{2}{\sqrt{\pi}} \left[1 + \frac{21}{16} V + O(V^2) \right]. \quad (22)$$

whence it follows that $\bar{a} \approx 0.9 \times 10^{-33}$ cm for $0 < V < 0.08$. The value \bar{a} determines the mean amplitude of oscillations of the classical universe filled with matter and radiation, thereby specifying its actual linear dimension. The maximal proper distance in a closed universe can be estimated at $d \approx \pi \bar{a} \approx 3 \times 10^{-33}$ cm; that is, the universe in the lowest state has a proper dimension on the order of the Planck length [19]. The presence of the minimal length removes the problem of the initial cosmological singularity.

Equations (7) and (8), which are obtained within the general theory of relativity, also dictate the relationship between the quantities a , ϵ and V . By using the value of $\epsilon_0 \simeq 2.6$, which we found for $V \simeq 0.08$, we can estimate the classical turning points at $a_1 \simeq 1.4 \times 10^{-33}$ cm and $a_2 \simeq 2.2 \times 10^{-33}$ cm. The value of a_1 determines the maximal dimension of the universe occurring in the lowest state to the left of the barrier along the a axis, while a_2 characterizes its initial dimension upon tunneling from this state. In

this era, the matter- and radiation- energy densities are $T_0^0 \approx V \approx 1.3 \times 10^{77} \text{ GeV/fm}^3$, and $\tilde{T}_0^0 \approx \frac{\epsilon_0}{\bar{a}^4} \approx 1.7 \times 10^{78} \text{ GeV/fm}^3$, respectively; that is, we can see that, according to our model, the energy density in the early universe is determined primarily by the radiation-energy density. This result is fully consistent with what is commonly thought about the properties of the universe for $a \rightarrow 0$ [9]. The total matter-energy density is $\rho \approx 0.64 m_p^4$, where m_p is the Planck mass. Thus, we can see that quite reasonable results are obtained when the parameters ϵ and V as derived on the basis of quantum theory are used in the equations of the general theory of relativity.

It is interesting to estimate the quantity n at the \bar{a} value coincident with the dimension of the presently observed part of the Universe. From the relation $\bar{a} \sim \sqrt{n}$ at $\bar{a} \sim 10^{28} \text{ cm}$, we obtain $n \sim 10^{122}$, whence we can see that, if the quantum model being considered is extrapolated to the observed Universe, it occurs in a highly excited state. Quantum corrections to the classical equations of the general theory of relativity are extremely small in this case (they are of order $\sim 1/n$). The resulting value of $n \sim 10^{122}$ is consistent with the estimates presented in [11, 21] and is confirmed by rigorous quantum-mechanical calculations within the radiation-free model that was considered in [19] and which is justified for the present, large, values of \bar{a} at the matter-dominated stage.

4. CONCLUSION

The presence of radiation in the universe makes it possible to associate a privileged reference frame with it and to remove thereby an ambiguity in choosing the time coordinate. This opens new possibilities both in classical and in quantum cosmology. Upon performing quantization, there naturally arises the Schrödinger equation (13) with an effective interaction U in the form of a potential barrier. The evolution of the universe involves a quantum stage that is realized in the prebarrier region and which precedes the process of tunneling through the barrier. The dynamics of this stage is governed by the interaction of the gravitational and scalar fields. That the system in question possesses a spectrum of quantum (quasistationary) states and that transitions can occur between these states enable us to take a fresh look at the problem of the dimension of the Universe. In the approach developed here, the universe is characterized by a minimal length, so that the singularity problem does not arise in it. The probability for the universe to undergo a tunnel transition is maximal in the lowest quantum state, where the energy density and the scale factor are on the same orders of magnitude as the corresponding Planck values. If a quantum universe tunnels from higher states, the dimensions of the region from which tunneling occurs can considerably exceed the Planck length. The constants E and V appearing in the Einstein equations are determined by the preceding, quantum stage. The use of the parameters in the general theory of relativity that were obtained on the basis of quantum theory leads to conclusions that are consistent with the currently adopted concepts of the early Universe and its subsequent evolution.

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